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Knot points of a double-covariant system of elliptic equations and preferred frames in general relativity

V Pelykh

Pidstryhach Institute of Applied Problems in Mechanics and Mathematics, Ukrainian National Academy of Sciences, 3B Naukova Str, Lviv, 79601, Ukraine

E-mail: pelykh@lms.lviv.ua

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Abstract

We investigate the elliptic system of equations, which is general-covariant and locally $SU(2)$-covariant. For this system, we obtain the new condition of the Dirichlet problem solvability and the condition of the absence of zeroes for the solutions. This system contains, in particular, the Sen–Witten equation. On this basis, we prove the existence of the wide class of hypersurfaces, in all points of which there exists a correspondence between the Sen–Witten spinor field and three-frames, as well as preferred lapses and shifts. The Nester special orthonormal frame also exists on a certain subclass containing not just the maximal hypersurfaces.

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1. Introduction

The necessity of investigating submanifolds, on which the solutions of elliptic equations are equal to zero, is connected with the fact that the necessary and sufficient conditions for the absence of such closed submanifolds of codimension one are simultaneously the necessary and sufficient conditions for uniqueness of the Dirichlet problem for these equations in the domain. Since the elliptic equations refer to the static solutions of the given hyperbolic field equations, the non-uniqueness of the solution for the boundary value problem defines the non-stability of 'zero modes' of given field equations. Additionally it appears to be necessary to study not only the closed submanifolds and not only of codimension one, but all other ones, on which zeros of solutions are located. This is related, in particular, to the Sen–Witten equation (SWE), for which the question about the existence of zeros has been discussed for a long time [1–4]. The absence of zeros for SWE solutions has been proven for the case when the initial data set for the Einstein equations on the maximal hypersurface is asymptotically flat, and the local mass condition is fulfilled [5]. Since on a maximal hypersurface the system of equations splits into separate equations, the choice of the Cauchy surface as maximal enables us to use the known results of investigations of zeros for single equations. We can also ascertain that, in each point of the maximal hypersurface, there exists a two-to-one correspondence between the Sen–Witten spinor and the Nester special orthonormal frame (SOF).

In this paper, we aim to develop a new approach for establishing the conditions of solvability and zeros absence for general, from the physical point of view, elliptic systems of equations. It will be possible to prove the existence of the wide class of hypersurfaces, in all points of which there exists the two-to-one correspondence between the Sen–Witten spinor and a certain three-frame, which we call the Sen–Witten orthonormal frame (SWOF). In all points on such hypersurfaces there are also well-defined lapses and shifts, associated by Ashteckar and Horowitz [1] with the Sen–Witten spinor. On a subclass of this class, including also the maximal hypersurfaces, we establish the existence of a two-to-one correspondence between the Sen–Witten spinor and the Nester three-frame.

2. Preliminaries

Firstly, we introduce three definitions.

Definition 1. *The knot point of the component of the solution is a point, in which the component is equal zero.*

Definition 2. *The knot point of the solution for the elliptical system of equations is a point, in which the solution is equal to zero.*

From the general theory of elliptic differential equations, it is known that nontrivial solutions cannot vanish on an open subdomain, but they can turn to zero on subsets of lower dimensions k, $k = 0, 1, \ldots, n - 1$, where n is the dimension of the domain.

Definition 3. *The knot submanifold of dimension* $s, s = 1, 2, \ldots, n - 1$, *is a maximal connected subset*[1] *of dimension s consisting of knot points of the solution.*

A discrete set of knot points is zero-submanifold. In section 3 we show that, for the system of differential equations in which we are interested, all knot subsets are formed by the intersection of knot surfaces of the components of the solution.

The connection between the unique solvability for the boundary value problem in $\mathbf{R}^n$ and the absence of $(n - 1)$-dimensional closed knot submanifolds was established by Picone [6, 7]. The existence of such a connection follows from the next consideration: if the boundary value problem in a certain domain Ω is uniquely solvable, then the boundary value problem is also uniquely solvable for any subdomain $\Omega_1 \subseteq \Omega$. This excludes the possibility of the existence of nontrivial solutions which become zero on the boundary of the arbitrary domain Ω_1, i.e. it excludes the possibility of the existence of closed knot submanifolds of codimension one, and vice versa.

The known investigations of the elliptical equations of a general form does not allow us to obtain the conditions for the absence of all knot points. For example, even in the case of the only single equation of a general form Aronszajn and Cordes proved, we prove the absence of zeros only of infinite order [8, 9]. This is why we further examine only such general equations, which possess also the necessary physical properties; in particular, symmetry properties.

Let Ω be a bounded closed spherical-type domain in three-dimensional Riemannian space V^3, otherwise: (i) its boundary $\partial\Omega$ in every point has a tangent plane; (ii) for every point P

[1] Maximal connected subset A is a nonempty connected subset such that the only connected subset containing A is A.

on the boundary there exists a sphere, which belongs to Ω, and the boundary of the sphere includes the point P.

In the domain Ω let us consider the system of elliptic second-order equations

$$\frac{1}{\sqrt{-h}}\frac{\partial}{\partial x^{\alpha}}\left(\sqrt{-h}h^{\alpha\beta}\frac{\partial}{\partial x^{\beta}}u_A\right)+C_A{}^{B}u_B=0 \tag{1}$$

where $h^{\alpha\beta}$ are components of the metric tensor in V^3. These are arbitrary real functions of independent real variables x^{α}, continuous in Ω, and the quadratic form $h^{\alpha\beta}\xi_{\alpha}\xi_{\beta}$ is negative definite. The unknown functions u_A of independent variables x^{α} are the elements of complex vector space $\mathbf{C}^2$, in which the skew symmetric tensor ε^{AB} is defined, and the group $SU(2)$ acts. $C_A{}^{B}$ is a Hermitian (1, 1) spinorial tensor.

The system of equation (1) is covariant under the arbitrary transformations of coordinates in V^3, and covariant under the local $SU(2)$-transformations of unknown functions in a local space isomorphic to the complexified tangent space in every point to V^3.

Picone ascertained that, at arbitrary coefficients of elliptic equations, the boundary value problem is uniquely solvable, and the closed knot submanifolds of codimension one are absent, respectively, only in the domains with a small enough intrinsic diameter.

The general conditions for the absence of closed knot surfaces for strong elliptic system (1) are ascertained by theorem 1 [10].

Theorem 1. *If in domain Ω there are symmetrical quadratic functional second-order matrices B_1, B_2, B_3 of C^1 class, such that matrix*

$$\sqrt{-h}C-\sum_{\alpha=1}^{3}\frac{\partial B_{\alpha}}{\partial x^{\alpha}}+B^{T}G^{-1}B$$

is positive definite, where $B=(B_1, B_2, B_3)$, $G=\sqrt{-h}\,\mathrm{diag}(\|h_{\alpha\beta}\|, \|h_{\alpha\beta}\|, \|h_{\alpha\beta}\|)$, then the solution of the system of equation (1) with matrix $C=\left\|C_A{}^{B}\right\|$ of C^1 class does not have the closed knot surfaces in domain Ω.

The effective geometrical conditions of the existence of the B-matrix and the corresponding unique solvability of the Dirichlet problem in dependence on the domain intrinsic diameter have been obtained [10] for Euclidean space. Such conditions are important, for example, in the theory of nuclear reactors. Since we are interested in the conditions of the absence of knot manifolds for quantum field equations, we further concentrate our attention on the conditions of the absence of knot points in the domains of arbitrary as well as infinite intrinsic diameters.

Evidently, if matrix C is positive definite, then the conditions of theorem 1 are fulfilled for $B\equiv 0$, and the closed knot surfaces are absent in the domain with an arbitrary intrinsic diameter. Simultaneously, the boundary value problem for the system of equation (1) is uniquely solvable.

Theorem 1 does not indicate the conditions at which the knot points, lines as well as all knot surfaces for the solution of equation (1), are absent. We obtain these in section 3.

3. Conditions for the absence of knot points

In the case of a single self-adjoint elliptic equation in V^3 the knot submanifolds can only be the surfaces which divide the domain, but in the case of a system of equations the topology of the knot submanifolds becomes more various; it can also be lines and points. We can take this fact into account and ascertain the conditions for the absence of knot manifolds exploiting the double covariance of the system of equation (1) and using the Zaremba–Giraud lemma, first generalized by Keldysh and Lavrentiev [11] and later by Oleynik [12].

Let us introduce the matrix

$$R := \left\| R_{A'}{}^{B} \right\| := \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} \qquad \alpha\overline{\alpha} + \beta\overline{\beta} = 1$$

which is of the group $SU(2)$, and let its elements additionally satisfy the condition

$$C_0{}^1\beta^2 + \left(C_0{}^0 - C_1{}^1\right)\alpha\beta - C_0{}^1\alpha^2 = 0.$$

Therefore

$$C_{0'}{}^{1'} = R_{0'}{}^{A} C_A{}^{B} R^{1'}{}_{B} = 0$$

and in accordance with the Hermiticity of matrix C also $\overline{C}_{0'}{}^{1'} = C_{1'}{}^{0'}$. Then $C_0 := C_{0'}{}^{0'}$ and $C_1 := C_{1'}{}^{1'}$ are eigenvalues of matrix $C = \left\| C_A{}^B \right\|$. This follows from the fact that, for arbitrary matrix $R \in SU(2)$, the identity

$$-\varepsilon R \varepsilon \equiv R^{T+}$$

is valid, where $\varepsilon = \|\varepsilon^{AB}\|$. Therefore

$$C' = -\varepsilon R \varepsilon C R^{T+} = R^{T+} C R^{T} = \mathrm{diag}(C_0, C_1).$$

Let us denote

$$\Delta := C_1{}^1 - C_0{}^0 - \left[\left(C_1{}^1 - C_0{}^0\right)^2 + 4\left|C_0{}^1\right|^2\right]^{1/2}$$

and let us denote by S a set of points in domain Ω, in all points of which $C_0{}^1$ does not equal zero. Also, let us denote by T a set of points, in which $C_0{}^1$ is equal to zero. Then the elements of the matrix R, which transforms the matrix C to diagonal form, satisfies on set S the conditions

$$\alpha\overline{\alpha}\left(1 + \Delta^2/4\left|C_0{}^1\right|^2\right) = 1 \qquad \beta = \alpha\Delta/2C_0{}^1$$

and on set T the conditions

$$\alpha\overline{\alpha} = 1 \qquad \beta = 0.$$

Functions $u_{0'}$ and $u_{1'}$ on set S are as follows

$$u_{0'} = \overline{\alpha}\left(u_0 + \frac{\Delta}{2\overline{C}_0{}^1}u_1\right) \qquad u_{1'} = \alpha\left(-\frac{\Delta}{2C_0{}^1}u_0 + u_1\right) \tag{2}$$

and on set T these are

$$u_{0'} = \overline{\alpha}u_0 \qquad u_{1'} = \alpha u_1. \tag{3}$$

Respectively, eigenvalue C_0 on S is

$$C_0 = \frac{4C_0{}^0\left|C_0{}^1\right|^4 + \left(4\Delta\left|C_0{}^1\right|^2 + C_1{}^1\Delta^2\right)\left(4\left|C_0{}^1\right|^2 + \Delta^2\right)}{4\left|C_0{}^1\right|^2\left(4\left|C_0{}^1\right|^2 + \Delta^2\right)}$$

and coincides with $C_0{}^0$ on set T.

Lemma 1. *If real and imaginary parts of functions u_A and of elements of matrix $C_A{}^B$ are functions of class C^2 in the domain Ω, then the real and imaginary parts of functions $u_{A'}$ defined by conditions (2)–(3) are also the functions of class C^2 in this domain.*

Proof. Taking into account that it is always possible to choose $\operatorname{Im}\alpha \in C^2(\Omega)$, from direct calculation we obtain that on set S there are first and second derivatives of real and imaginary parts of functions $u_{A'}$ and α with respect to arguments $\left(\Delta^2/4|C_0{}^1|^2\right)$ and $\left(\Delta/2|C_0{}^1|^2\right)$. Also

$$\lim_{P\ni S\to Q\in T} \operatorname{Re}\alpha^{(m)}(P) = \operatorname{Re}\alpha^{(m)}(Q) \qquad \lim_{P\ni S\to Q\in T} \operatorname{Im}\alpha^{(m)}(P) = \operatorname{Im}\alpha^{(m)}(Q)$$

$$\lim_{P\ni S\to Q\in T} \operatorname{Re} u_{A'}^{(m)}(P) = \operatorname{Re} u_{A'}^{(m)}(Q) \qquad \lim_{P\ni S\to Q\in T} \operatorname{Im} u_{A'}^{(m)}(P) = \operatorname{Im} u_{A'}^{(m)}(Q)$$

where symbol $f^{(m)}$ denotes the arbitrary partial derivatives of order $m = 0, 1, 2$. □

The following theorem is valid.

Theorem 2. *Let*

(a) real and imaginary parts of the elements of matrix C be of C^2 class in the domain Ω;
(b) at least one eigenvalue of matrix C, for definiteness C_0, is non-negative everywhere in Ω;
(c) real or imaginary parts of the function

$$v := \begin{cases} \left.\left(u_0 + \dfrac{\Delta}{2\overline{C}_0{}^1}u_1\right)\right|_{S\bigcap\partial\Omega} \\ u_0|_{T\bigcap\partial\Omega} \end{cases}$$

do not equal zero in any point.
Then solution u_A of class C^2 for the system of equation (1) does not have any knot points in the domain Ω of spherical type.

Proof. The system of equations (1) is covariant under the arbitrary transformations of coordinates and under the local transformations from the group $SU(2)$ that allows us to use them independently. As a first step, let us apply the $SU(2)$ spinor transformation $u_A \to R_{A'}{}^B u_B$, which transforms the matrix C to a diagonal form, and under which equation (1) is covariant.

The eigenvalues of matrix C are real, therefore the resulting system of equations (1) splits into a system of four independent equations for real and imaginary parts of spinor $u_{A'}$. Taking into account that $u_{A'}$, C_0 and C_1 are scalars under transformations of coordinates, and $C_0 \geqslant 0$, we can apply the Zaremba–Giraud principle in the general form grounded by Oleynik [12] to every equation containing C_0. According to this principle, if in a certain point P_0 on the sphere the nonconstant function in the ball turns to zero, and everywhere in the ball $\operatorname{Re} u_{0'} < 0$, then $\langle \mathrm{d}\operatorname{Re} u_{0'}, l\rangle_{|P_0} < 0$. Here l is the arbitrary vector field, for which $\langle n, l\rangle_{|P_0} > 0$, and n is the intrinsic one-form normal to the sphere in the point P_0.

Let us show further that a set of knot points for function $\operatorname{Re} u_{0'}$ does not contain the isolated points. Let us assume that such a point exists, i.e. $\operatorname{Re} u_{0'} = 0$, and in a certain neighbourhood of the point P_0 the function has a constant sign. For definiteness, in this neighbourhood let $u_{0'} < 0$. Let us consider a sphere, on which the point P lies, and it is so small that it completely belongs to the mentioned neighbourhood of the point P. Then, using the Zaremba–Giraud principle, we obtain $\langle \mathrm{d}\operatorname{Re} u_{0'}, n\rangle|_{P_0} > 0$. Therefore, in any neighbourhood of the point P_0, located outside the ball, the function $\operatorname{Re} u_{0'}$ changes its sign; this is why its zeros are not isolated. Therefore, they form the surfaces which divide Ω. Since $C_0 \geqslant 0$, then it follows from the maximum principle that the closed knot surfaces for the components of solution $\operatorname{Re} u_{0'}$ are absent. The analogous conclusion is true also for the component of solution $\operatorname{Im} u_{0'}$. This means that the only surfaces having common points with the boundary of domain Ω can be the knot surfaces of real or imaginary parts of function $u_{0'}$. According to condition (c), if, for definiteness,

$$\mathrm{Re}\left(u_0 + \frac{\Delta}{2\overline{C}_0{}^1}u_1\right)\bigg|_{S\bigcap\partial\Omega} \neq 0 \qquad \mathrm{Re}\, u_0|_{T\bigcap\partial\Omega} \neq 0$$

then we can choose

$$\mathrm{Re}\,\overline{\alpha}|_{S\bigcap\partial\Omega} \neq \left\{\left[\mathrm{Re}\left(u_0 + \frac{\Delta}{2\overline{C}_0{}^1}u_1\right)\right]^{-1} \mathrm{Im}\,\overline{\alpha}\,\mathrm{Im}\left(u_0 + \frac{\Delta}{2\overline{C}_0{}^1}u_1\right)\right\}\bigg|_{S\bigcap\partial\Omega}$$

$$\mathrm{Re}\,\overline{\alpha}|_{T\bigcap\partial\Omega} \neq [(\mathrm{Re}\,u_0)^{-1}\,\mathrm{Im}\,\overline{\alpha}\,\mathrm{Im}\,u_0]|_{T\bigcap\partial\Omega}$$

and we obtain

$$\left[\mathrm{Re}\,\overline{\alpha}\,\mathrm{Re}\left(u_0 + \frac{\Delta}{2\overline{C}_0{}^1}u_1\right) - \mathrm{Im}\,\overline{\alpha}\,\mathrm{Im}\left(u_0 + \frac{\Delta}{2\overline{C}_0{}^1}u_1\right)\right]\bigg|_{S\bigcap\partial\Omega} \equiv \mathrm{Re}\,u_{0'}|_{S\bigcap\partial\Omega} \neq 0$$

$$(\mathrm{Re}\,\overline{\alpha}\,\mathrm{Re}\,u_0 - \mathrm{Im}\,\overline{\alpha}\,\mathrm{Im}\,u_0)\,|_{T\bigcap\partial\Omega} \equiv \mathrm{Re}\,u_{0'}|_{T\bigcap\partial\Omega} \neq 0.$$

Therefore, knot surfaces as well as lines and points of the real (or imaginary) parts are absent. This is why any knot points of the complete solution u_A are also absent. The statement of the theorem is proven. □

Note that, if the conditions (a) and (b) of the theorem are fulfilled, and if the matrix C is non-negative definite in domain Ω, then both eigenvalues are non-negative and, therefore, the boundary value problem for the system of equations (1) is uniquely solvable in the arbitrary bounded domain, as it follows from the classical maximum principle. Otherwise, the solution in the finite domain exists only when its intrinsic diameter does not overcome a certain value.

4. The conditions of the absence of knot points for the solutions of the SWE

After Witten's positive energy proof, the attempts to develop the tensor method for the proof were performed in two ways. Firstly, there were attempts to interpret the tensor for the Sen–Witten spinor field. In particular, Ashteckar and Horowitz [1] used the Sen–Witten spinor field to determine a class of preferred lapses $T := \lambda$ and shifts $T^a := -\sqrt{2}\mathrm{i}\lambda^{+(A}\lambda^{B)}$. Dimakis and Müller-Hoissen [2, 3] defined a preferred class of orthonormal frame fields in which the spinor field takes a certain standard form. Frauendiener [13] noticed a correspondence between the Sen–Witten spinor field and a triad. But, as shown by Dimakis and Müller-Hoissen, frame fields cannot exist in the knot points of the spinor field.

Among the works performed in the second way, the most developed is Nester's method which is grounded on the new gauge conditions for the SOF

$$*\,q := \varepsilon^{abc}\omega_{abc} = 0 \qquad \widetilde{q}_b := \omega^a{}_{ba} = F_b \tag{4}$$

where ω_{abc} are the connection one-form coefficients and F is arbitrary everywhere on the Σ_t defined exact one-form.

An essential part of Witten's proof of non-negativity for ADM mass is the application of the SWE

$$\mathcal{D}^B{}_C\beta^C = 0 \tag{5}$$

with appropriate asymptotic conditions on the space-like hypersurface Σ in four-dimensional Riemannian manifold $M = \Sigma \times R$ with each $\Sigma_t = \Sigma \times \{t\}$ space-like. The initial data set $(\Sigma_t, h_{\mu\nu}, \mathcal{K}_{\pi\rho})$ satisfies the constraints, and is asymptotically flat. An action of the operator

$\mathcal{D}_{AB}$ on spinor fields is

$$\mathcal{D}_{AB}\lambda_C = D_{AB}\lambda_C + \frac{\sqrt{2}}{2}\mathcal{K}_{ABC}{}^D\lambda_D$$

where D_{AB} is the spinorial form of the derivative operator D_α compatible with the metric $h_{\mu\nu}$ on the C^∞ hypersurface Σ_t, and $\mathcal{K}_{ABCD}$ is the spinorial tensor of the extrinsic curvature of hypersurface Σ_t.

The existence and uniqueness theorem for the solution of equation (5) in corresponding Hilbert space with some asymptotic conditions was proven by Reula [14] (see also [1]).

Let us ascertain the conditions of the absence of zeros for these solutions on Σ_t using the results of section 3. From equation (5), taking into account the equation of Hamiltonian constraint on Σ_t, in Gaussian normal coordinates we obtain [5]

$$\begin{aligned}\mathcal{D}_A{}^B\mathcal{D}_{BC}\lambda^C &= \frac{1}{2\sqrt{-h}}\frac{\partial}{\partial x^\alpha}\left(\sqrt{-h}h^{\alpha\beta}\frac{\partial}{\partial x^\beta}\lambda_A\right) - \frac{\sqrt{2}}{2}\mathcal{K}D_{AB}\lambda^B \\ &\quad - \frac{\sqrt{2}}{4}\lambda^B D_{AB}\mathcal{K} + \frac{1}{4}\mathcal{K}^2\lambda_A + \frac{1}{8}\mathcal{K}_{\alpha\beta}\mathcal{K}^{\alpha\beta}\lambda_A + \frac{1}{4}\mu\lambda_A = 0. \qquad (6)\end{aligned}$$

Therefore, the system of equations (6) is a system of the form (1); if it does not have the knot points, the SWE also does not have them.

The spinorial tensor

$$C_A{}^B := \frac{\sqrt{2}}{4}D_A{}^B\mathcal{K} + \frac{1}{4}\varepsilon_A{}^B\left(2\mathcal{K}^2 + \frac{1}{2}\mathcal{K}_{\pi\rho}\mathcal{K}^{\pi\rho} + \mu\right) \qquad (7)$$

is Hermitian because $\left(\mathcal{D}_A{}^B\mathcal{K}\right)^+ = (\varepsilon^{BC}\mathcal{D}_{AC}\mathcal{K})^+ = (\varepsilon^{BC})^+(\mathcal{D}_{AC}\mathcal{K})^+ = -(\mathcal{D}_{AC}\mathcal{K})\varepsilon^{CB} = (\mathcal{D}_A{}^B\mathcal{K})$.

So, the SWE solutions of class C^2 do not have the knot points in a bounded closed domain Ω of spherical type on Σ_t, if for the spinorial tensor $C_A{}^B$ in this domain and for the boundary values of the solution the conditions of theorem 2 are fulfilled.

Furthermore, let us consider a sequence Ω_n of increasing domains of spherical type covering Σ_t. If in every domain the conditions of theorem 2 are fulfilled, then none of the solutions of class C^2 have knot points in Ω_n. According to Reula, on Σ_t there exists the SWE solution of the $\lambda^C = \lambda^C_\infty + \beta^C$ form, where λ^C_∞ is the asymptotically covariant constant spinor field on Σ_t. β^C is an element of Hilbert space $\mathcal{H}$, which is the Cauchy completion of C_0^∞ spinor fields under the norm

$$\|\beta^E\|^2_{\mathcal{H}} = \int_{\Sigma_t}\left(\mathcal{D}^A{}_B\beta^B\right)^+\left(\mathcal{D}_{AC}\beta^C\right)\mathrm{d}V.$$

The solution λ^C belongs properly to the C^∞ class. From the asymptotic flatness condition it follows that $\left(\Delta^2/\left(4\left|C_0{}^1\right|^2\right)\right.$, as well as real and imaginary parts of functions $\left(\Delta/2\overline{C}_0{}^1\right)$ and $\left(\Delta/2C_0{}^1\right)$, vanish asymptotically. Therefore, condition (c) from theorem 2 asymptotically takes the form, $\operatorname{Re}\lambda^0_\infty \neq 0$ or $\operatorname{Im}\lambda^0_\infty \neq 0$. In such a way we obtain the following theorem.

Theorem 3. *Let*

(a) initial data set be asymptotically flat;

(b) everywhere on Σ_t the matrix of the spinorial tensor (7) has at least one non-negative eigenvalue, for definiteness C_0;

(c) $\operatorname{Re}\lambda^0_\infty$ or $\operatorname{Im}\lambda^0_\infty$ asymptotically nowhere equal to zero.

Then the asymptotically constant nontrivial solution λ^C to the SWE does not have the knot points on Σ_t.

The conditions of theorem 3 are fully admissible from the physical point of view.

5. Towards the SWE: special orthonormal frame and preferred time variables

Usually the question about the existence of a system of coordinates or an orthonormal basis, which satisfy certain gauge conditions, is reduced to the question about the existence of a solution for nonlinear system of differential equations. It can often be solved only with some additional limitations and assumptions [15].

The existence theorem for the SWE (linear) and theorem 3 about zeros (section 4) on surfaces, which cannot be maximal, allow us to prove the existence of a certain class of orthonormal three-frame in all points of these hypersurfaces which satisfies gauge conditions

$$\begin{aligned} &\varepsilon^{abc}\omega_{abc} \equiv *q = 0 \qquad && \omega^{a}{}_{1a} \equiv -\widetilde{q}_1 = F_1 \\ &\omega^{a}{}_{2a} = -\widetilde{q}_2 = F_2 \qquad && \omega^{a}{}_{3a} = -\widetilde{q}_3 = \mathcal{K} + F_3 \end{aligned} \tag{8}$$

and generalizes Nester's SOF. We call such a three-frame the SWOF.

Theorem 4. *Let the conditions of theorem 3 be fulfilled. Then everywhere on Σ_t there exists a two-to-one correspondence between the Sen–Witten spinor and the SWOF.*

Proof. In reality, let all conditions of theorem 3 be fulfilled on Σ_t. Then the SWE solution λ_A does not have knot points anywhere on Σ_t. This allows us to prove on such Σ_t the Sommers [16] assumption that the spatial null one-form $L = -\lambda_A\lambda_B$ on Σ_t is nonzero. It allows us to turn everywhere on Σ_t to the 'squared' SWE represented in the following form

$$\langle \widetilde{L}, D \otimes L\rangle - \mathcal{K}L + 3!\mathrm{i} * (n \wedge D \wedge L) = 0 \tag{9}$$

where $\langle \widetilde{L}, D \otimes L\rangle$ is the one-form with components $\widetilde{L}_\nu D_\mu L^\nu$, $\widetilde{L} = |L|^{-1} * (L \wedge \overline{L})$ is the nonzero spatial one-form, and n is the one-form of a unit normal to Σ_t.

The bilinear form

$$\frac{1}{\sqrt{2}} n^{A\dot{A}}\lambda_A\overline{\lambda}_{\dot{A}} = \lambda_A\lambda^{A+} \equiv \lambda$$

where n is the one-form of a unit normal to Σ_t, is Hermitian positive definite, and the solution λ_A does not have knot points on Σ_t. Consequently, we can further introduce the real nowhere degenerated orthonormal four-coframe θ^m as

$$\theta^0 \equiv n = N\,\mathrm{d}t \qquad \theta^1 = \frac{\sqrt{2}}{2\lambda}(L + \overline{L}) \qquad \theta^2 = \frac{\sqrt{2}}{2\lambda\mathrm{i}}(L - \overline{L}) \qquad \theta^3 = \widetilde{L} \tag{10}$$

and represent immediately equation (9) in the form

$$-\langle \theta^1, D \otimes \theta^3\rangle - \mathcal{K}\theta^1 + 3! * [n \wedge (D + F) \wedge \wedge\theta^2] = 0 \tag{11}$$

$$\langle \theta^2, D \otimes \theta^3\rangle + \mathcal{K}\theta^3 + 3! * [n \wedge (D + F) \wedge \theta^1] = 0 \tag{12}$$

where $F = D\ln\lambda$. The system of equations (11) and (12) includes only four independent equations, and these are equations (8) for the connection one-form coefficients. From this, it follows that, if on Σ_t the conditions of theorem 3 and the SWE are fulfilled, then on Σ_t there exists the three-frame θ^a defined by equations (10) in which conditions (8) are fulfilled.

Inversely, if on Σ_t in some three-frame θ^a the conditions of theorem 3 and conditions (8) are fulfilled, then it follows from the condition of theorem 3 that these one-forms have a form $\theta^a = \theta^a_\infty + \phi^a$, where θ^a_∞ tend asymptotically to the covariant constant forms and ϕ^a belongs to $\mathcal{H}$. We can turn from four-frame $\theta^m \equiv \{n, \theta^a\}$ to one-forms $\theta^0, L, \widetilde{L}$, assuming $\lambda_A|_{\Sigma_t} \neq 0$. After this, we obtain equation (9) and furthermore (5)[2] for the spinor field λ^A,

[2] The equivalence of the SWE (5) and of the equation (6) is proven by Reula [14].

which, as we have demonstrated previously, indeed does not have knot points on selected hypersurface Σ_t and which, together with asymptotical conditions, defines up to sign the spinor field λ^A. Mentioned in the conditions of the theorem, the correspondence between the Sen–Witten spinor field and Nester's SOF is defined by the relationship (10). □

We have proven [5] that if the initial set $(\Sigma_t, h_{\mu\nu}, \mathcal{K}_{\pi\rho})$ on the maximal hypersurface[3] Σ_t is asymptotically flat and satisfies the dominant energy condition, then everywhere on Σ_t from the existence of the Sen–Witten spinor field follows the existence of Nester's three-frame. Conversely theorem 3 allows us to strengthen significantly this result by taking away the assumption that Σ_t is maximal. Indeed, if all the conditions of theorem 3 are fulfilled on Σ_t, and additionally the one-form $\mathcal{K}\widetilde{L}$ is globally exact, we can perform in these conditions the identification $F \equiv \mathrm{d}\ln\lambda + \mathcal{K}\theta^3$ and obtain the Nester's gauge (4), or we can perform the inverse transition, i.e. from Nester's gauge to SWE. Therefore, if on Σ_t the conditions of theorem 3 are fulfilled, then the SWE and Nester's gauge are equivalent if and only if the one-form $\mathcal{K}\lambda^{+(A}\lambda^{B)}$ is exact. In this case the correspondence between the Sen–Witten spinor and Nester's SOF is also ascertained by relationship (10).

Ashteckar and Horowitz [1] have emphasized the necessity of investigating zeros for SWE solutions, introducing the vector interpretation of the Sen–Witten spinor, which defines a preferred lapse and shift. Evidently, the fulfilment of the conditions of theorem 3 ensures the existence of corresponding lapses and shifts well defined everywhere on Σ_t. Also, the preferred class of orthonormal four-frame fields introduced by Dimakis and Müller-Hoissen exists in all points of Σ_t under the fulfilment of the conditions of theorem 3.

6. Conclusions

The presence of zeros in the solutions of elliptic equations is ordinary rather than exceptional. Therefore, it is necessary to prove the absence of zeros for concrete cases.

The represented investigation demonstrates the possibility of obtaining the condition of the absence of knot manifolds for a general enough system of elliptic second-order equations owing to its double covariance.

The application of this result to the SWE allows us to prove the equivalence of the SWE and gauge conditions (8) and, respectively, the existence of an everywhere well-defined two-to-one correspondence between the Sen–Witten spinor field and the SWOF, which is the Nester SOF in the particular case when one of the one-forms $\mathcal{K}\theta^a$ is exact. Therefore, the indicated correspondence exists not only on the unique—maximal—hypersurface, but on the whole set of asymptotically flat hypersurfaces.

Ashteckar and Horowitz [1] have shown that the Reula results hold even if the energy condition is mildly violated. Also, the conclusion about the existence of special three-frames and four-frames, as well as preferred lapses and shifts, is stable under the violation of the energy condition because, as seen from equation (7), there exist hypersurfaces on which this condition of the absence of knot points is fulfilled with the violation of the energy condition.

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